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Dynamic behavior and modal control of beams under moving mass

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Abstract

The constitutive equation of an Euler–Bernoulli beam under the excitation of moving mass is considered. The dynamics of the uncontrolled system is governed by a linear, self-adjoint partial differential equation. A Dirac-delta function is used to describe the position of the moving mass along the beam and its inertial effects. An approximate formulation to the problem is obtained by limiting the inertial effect of the moving mass merely to the vertical component of acceleration. Having defined a ‘‘critical velocity’’ in terms of the fundamental period and span of the beam, it is shown that for smaller velocities, the approximate and exact approaches to the problem almost coincide. Since, the defined critical velocity is fairly large compared to those in practical cases, the approximate approach can effectively be used for a wide range of problems. There is however a slight variation in critical velocity depending on the weight of the moving mass. On the other hand, it is shown that the effect of higher vibrational modes is not negligible for certain velocity ranges. Finally, a linear classical optimal control algorithm with a time varying gain matrix with displacement-velocity feedback is used to control the response of the beam. The efficiency of the control algorithm in suppressing the response of the system under the effect of moving mass with different number of controlled modes and actuators is investigated.

1. Introduction

The dynamic behavior of the structures under the influence of moving loads is a subject of considerable engineering importance. A large amount of literature has been produced by various researchers and engineers on this regard in the last few decades. There are clearly many problems of great physical significance in which the load inertia is not negligible and can significantly alter the dynamic behavior of the system. Such systems are considered by many researchers such as Iwan and Stah [1], Stanisic [2], Akin and Mofid [3], Mahmoud and AbouZaid [4]. Comprehensive studies on the dynamic stability of continuous systems under the effect of inertial forces and high velocities of moving loads have been provided by Kononov and Borst [5], and Verichev and Metrikine [6]. Rao [7] has studied the behavior of a simply supported Euler–Bernoulli beam

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under the effect of a moving load while the inertial effect is included in the analysis. He considered the effects of centripetal and Coriolis accelerations besides the vertical component of acceleration of the moving mass. Using the mode superposition and multiple scale method, he showed that the effect of moving mass on the transient response of the beam is considerable. Bilello et al. [8] investigated the effect of moving mass on a prototype single-span bridge structure by conducting an experimental simulation of a small-scale model. The results of the experiments were compared to that of an analytical model using eigenfunction expansion approach for the case of moving mass and moving load. It was shown that there is a good correlation between the analytical and experimental results, without observing any significant inertial effects for the range of mass velocities considered in the study.

The active structural control on the other hand, has emerged as a new tool for suppressing the dynamic response of continuous or discrete flexible structures in recent years. A valuable literature survey on this subject has been provided by Meirovitch [9]. Many control algorithms and implementation mechanisms have been proposed by researchers on this regard, e.g. Yang et al. [10], Tadjbakhsh and Rofooei [11], Rofooei and Monajemi-Nejad [12]. Also, the application of smart materials as a new means for vibration suppression of continuous structural systems has been heavily explored. Sung [13] has studied the active control of a simply supported beam under a moving mass using piezo materials as actuators. He used a classical optimal control algorithm with displacement-velocity feedback, while the optimal placements of piezoactuators are determined by the minimization of an appropriate cost function. The results showed a good performance for the selected control algorithms.

In this work, the governing differential equation of motion for an Euler–Bernoulli beam under moving mass is employed taking into account the centripetal and Coriolis accelerations besides the vertical component one. Also, an approximate formulation to the problem is obtained merely by retaining the vertical component of acceleration, as it is suggested in a number of previous works [2,3,14]. Defining a critical velocity in terms of the fundamental period and span of the beam, the accuracy of the approximate method in estimating the dynamic response of the system is examined using different weight and velocities for the moving mass. It is shown that for velocities below the critical velocity, the approximate and exact approaches of the problem almost coincide. Since, the defined critical velocity is fairly large compared to those in practical cases, the approximate approach can effectively be applied instead of exact approach for a wide range of problems.

For masses moving with velocities higher than the critical one, exact formulation of the problem will be required. It should be noted that there is a slight variation in critical velocity depending on the weight of the moving mass. On the other hand, in case of moving mass, it is shown that the effect of higher vibrational modes is not negligible, especially for the velocities above a specific level.

Finally, a linear classical optimal control algorithm with a time varying gain matrix with displacement-velocity feedback is used to control the response of the beam under the effect of moving mass. Parametric studies are carried out to explore the effect of controlling the higher vibrational modes and the number of actuators on the performance of the control system. The results are indication of applicability of the active structural control in reducing the system's dynamic response.

2. Problem formulation

A uniform Euler–Bernoulli beam with moving mass and various boundary conditions is considered. A moving mass M whose inertia is assumed to be relatively large, travels along the beam with a constant velocity. The bending stiffness, EI and the mass per unit length of the beam m, is assumed to be constant. Let $z(x, t)$ denote the deflection of the beam with x and t representing the position of a point in the domain D and the time, respectively. For a one-dimensional beam problem, the position vector x will represent the x-axis with its origin coincided on the left support. The initial conditions for this undamped dynamic system are considered to be $z(x, 0) = g_1(x)$ and $\partial z(x,0)/\partial t = g_2(x)$ in which $g_1(x)$ and $g_2(x)$ are any continuous functions. Also, assume that n actuators are used in the system at locations x_i, $i = 1,2,3,\ldots, n$ to exert the required control forces $u_i(t)$, as it is shown in Fig. (1). The equation of motion of the controlled system can be described by

$$m\ddot{z} + Lz = f(x,t) + \sum_{i=1}^{n} u_i(t)\delta(x - x_i), \quad x \in D, \quad t > t_0, \tag{1}$$

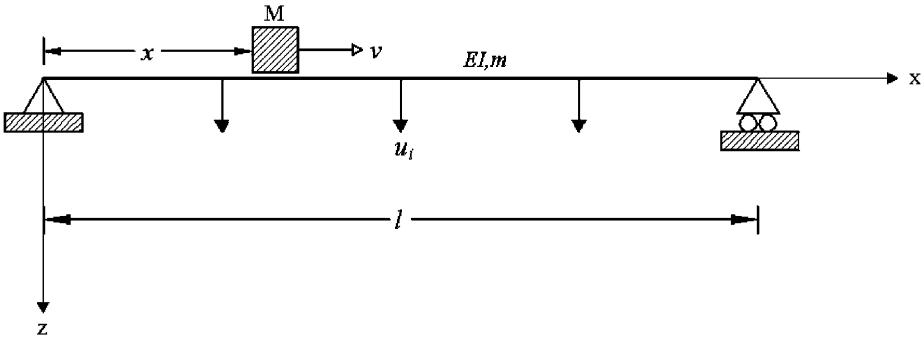

Fig. 1. Schematic figure of the dynamic system.

where

$$L = \frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2}{\partial x^2}\right] = EI\frac{\partial^4}{\partial x^4}$$

and

$$f(x,t) = M\left(g - \frac{\mathrm{d}^2 z_0(t)}{\mathrm{d}t^2}\right)\delta(x - vt) = M\left(g - \frac{\mathrm{d}^2 z(vt,t)}{\mathrm{d}t^2}\right)\delta(x - vt)$$
$$= M\left(g - \frac{\partial^2 z}{\partial t^2} - 2v\frac{\partial^2 z}{\partial x \partial t} - v^2\frac{\partial^2 z}{\partial x^2}\right)_{x=vt}\delta(x - vt). \tag{2}$$

The $z_0(t)$ denotes the vertical displacement of the mass M, that is equal to $z(x = vt,\ t)$ assuming continuous contact between the mass and the beam. It should be noted that in the approximate approach to the problem, only the vertical component of acceleration, $(\partial^2 z/\partial^2 t)$ is retained in the formulation, as it is suggested in a number of previous works [2,3,14]. Also, u_i and v are the control force and the velocity of the mass M, respectively, while g is the acceleration of gravity.

Using the eigenfunction expansion, the displacement z, and the forcing function f can be represented as $z = \sum_{k=1}^{\infty}\phi_k(x)A_k(t)$, and $f(x,t) + \sum_{i=1}^{n}u_i(t)\delta(x - x_i) = \sum_{k=1}^{\infty}\phi_k(x)B_k(t)$ in which $A_k(t)$ and $B_k(t)$ are time-dependent modal amplitudes and ϕ_k are the beam's orthogonal modal shapes [9]. Considering the first p vibrational modes, the right-hand side of Eq. (1) can be re-written as

$$M\left(g - \frac{\partial^2 z}{\partial t^2} - 2v\frac{\partial^2 z}{\partial x \partial t} - v^2\frac{\partial^2 z}{\partial x^2}\right)_{x=vt}\delta(x - vt) + \sum_{i=1}^{n}u_i(t)\delta(x - x_i) = \sum_{i=1}^{p}\phi_i(x)B_i(t). \tag{3}$$

Substitution of the above expansion in Eq. (3) results in:

$$M\left(g - \sum_{i=1}^{p}\left[\phi_i(x)\frac{\mathrm{d}^2 A_i}{\mathrm{d}t^2}(t) + 2v\phi_i'(x)\frac{\mathrm{d}A_i}{\mathrm{d}t}(t) + v^2\phi_i''(x)A_i(t)\right]\right)\delta(x - vt)$$
$$+ \sum_{i=1}^{n}u_i(t)\delta(x - x_i) = \sum_{i=1}^{p}\phi_i(x)B_i(t). \tag{4}$$

Multiplying both sides of Eq. (4) by $\phi_k(x)$ and integrating for the entire beam yields:

$$M\left(g - \sum_{i=1}^{p}\left[\phi_i(vt)\frac{\mathrm{d}^2 A_i}{\mathrm{d}t^2}(t) + 2v\phi_i'(vt)\frac{\mathrm{d}A_i}{\mathrm{d}t}(t) + v^2\phi_i''(vt)A_i(t)\right]\right)\phi_k(vt)$$
$$+ \sum_{i=1}^{n}u_i(t)\phi_k(x_i) = V_k B_k(t), \tag{5}$$

where

$$V_k = \int_0^l \phi_k^2(x)\,\mathrm{d}x. \tag{6}$$

So

$$B_k(t) = \left(\frac{M}{V_k}\right)\left(g - \sum_{i=1}^{p}\left[\phi_i(vt)\frac{\mathrm{d}^2A_i}{\mathrm{d}t^2}(t) + 2v\phi_i'(vt)\frac{\mathrm{d}A_i}{\mathrm{d}t}(t) + v^2\phi_i''(vt)A_i(t)\right]\right)\phi_k(vt) + \left(\frac{1}{V_k}\right)\sum_{i=1}^{n} u_i(t)\phi_k(x_i). \tag{7}$$

The modal amplitudes $A_k(t)$ can be determined by substituting Eqs. (7) and (3) into Eq. (1):

$$m\sum_{i=1}^{p}\phi_i\frac{\mathrm{d}^2A_i}{\mathrm{d}t^2} + EI\sum_{i=1}^{p}A_i\phi_i^{iv} - \sum_{i=1}^{p}\frac{\phi_i}{V_i} \times \left\{\sum_{j=1}^{n}u_j(t)\phi_i(x_j) + M\left(g - \sum_{k=1}^{p}\left[\phi_k(vt)\frac{\mathrm{d}^2A_k}{\mathrm{d}t^2}(t) + 2v\phi_k'(vt)\frac{\mathrm{d}A_k}{\mathrm{d}t}(t) + v^2\phi_k''(vt)A_k(t)\right]\right)\phi_i(vt)\right\} = 0. \tag{8}$$

Also, considering Eq. (8), one can conclude that:

$$\sum_{i=1}^{p}\phi_i\left\{m\frac{\mathrm{d}^2A_i}{\mathrm{d}t^2} + m\omega_i^2A_i - \frac{1}{V_i}\left(\sum_{j=1}^{n}u_j(t)\phi_i(x_j)\right) + \frac{M}{V_i}\left[g - \sum_{k=1}^{p}\left(\frac{\mathrm{d}^2A_k}{\mathrm{d}t^2}\phi_k(vt) + 2v\phi_k'(vt)\frac{\mathrm{d}A_k}{\mathrm{d}t} + v^2\phi_k''(vt)A_k\right)\right]\phi_i(vt)\right\} = 0 \tag{9}$$

in which

$$m\omega_j^2 = \int_0^l \phi_j L\phi_j\,\mathrm{d}x, \quad j = 1,2,\ldots. \tag{10}$$

Eq. (9) must be satisfied for any arbitrary ϕ_i. This is possible only when the expression in the bracket be equal to zero:

$$m\frac{\mathrm{d}^2A_i}{\mathrm{d}t^2} + m\omega_i^2A_i - \frac{1}{V_i}\left\{\sum_{j=1}^{n}u_j(t)\phi_i(x_j) + M\left[g - \sum_{k=1}^{p}\left(\frac{\mathrm{d}^2A_k}{\mathrm{d}t^2}\phi_k(vt) + 2v\phi_i'(vt)\frac{\mathrm{d}A_k}{\mathrm{d}t} + v^2\phi_i''(vt)A_k\right)\right]\phi_i(vt)\right\} = 0, \;\; i = 1,2,\ldots. \tag{11}$$

Eq. (11) is a set of p coupled ordinary differential equations. Using orthonormal system of eigenfunctions ($V_k = 1$, $k = 1,2,\ldots$) and re-arranging them in a matrix form results in:

$$\mathbf{T}(t)\ddot{\mathbf{A}} + \mathbf{Y}(t)\dot{\mathbf{A}} + \mathbf{\Lambda}(t)\mathbf{A} = \mathbf{f}(t) + \mathbf{B}\,\mathbf{u}(t), \quad \mathbf{A}(t_0) = \mathbf{A}_0, \;\; \dot{\mathbf{A}}(t_0) = \mathbf{B}_0, \tag{12}$$

in which $\mathbf{A}_0, \mathbf{B}_0$ and $\mathbf{f}$ are p-dimensional vectors with their elements defined as

$$(A_{0j}, B_{0j}) = \int_0^l m(z_0, \dot{z}_0)(\phi_j/m)\,\mathrm{d}x,$$

$$f_j = \int_0^l Mg\delta(x - vt)(\phi_j/m)\,\mathrm{d}x, \quad j = 1,2,3\ldots. \tag{13}$$

The elements of the matrix $\mathbf{B}_{p\times n}$ are given by:

$$B_{ji} = \phi_j(x_i)/m, \quad i = 1,2,3,\ldots,n. \tag{14}$$

$\mathbf{T}(t)$, $\mathbf{Y}(t)$ and $\mathbf{\Lambda}(t)$, are also $p \times n$ matrixes with their elements defined as:

$$T_{ij} = \delta_{ij} + \frac{M}{m}\phi_i(vt)\phi_j(vt), \tag{15}$$

$$Y_{ij} = \frac{2vM}{m}\phi_i(vt)\phi'_j(vt), \tag{16}$$

$$\Lambda_{ij} = \omega_j^2\delta_{ij} + \frac{v^2M}{m}\phi_i(vt)\phi''_j(vt). \tag{17}$$

Multiplying Eq. (12) by $\mathbf{T}(t)^{-1}$, the state-space form of this equation can be shown as:

$$\dot{\mathbf{X}}(t) = \bar{\mathbf{A}}(t)\mathbf{X}(t) + \bar{\mathbf{D}}(t)\,\mathbf{u}(t) + \bar{\mathbf{E}}(t)\,\mathbf{f}(t), \tag{18}$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} \\ \dot{\mathbf{A}} \end{bmatrix}_{2p\times 1}, \bar{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{T}^{-1}\mathbf{\Lambda} & -\mathbf{T}^{-1}\mathbf{Y} \end{bmatrix}_{2p\times 2p}, \bar{\mathbf{D}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix}_{2p\times 2p} \begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}_{2p\times n}$$

$$\bar{\mathbf{E}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix}_{2p\times 2p} \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix}_{2p\times p}, \mathbf{X}(t_0) = \begin{bmatrix} \mathbf{A}(t_0) \\ \dot{\mathbf{A}}(t_0) \end{bmatrix}_{2p\times 1}.$$

The state vector $\mathbf{X}(t)$ can be obtained as follows [15]:

$$\mathbf{X}(t) = \mathbf{U}(t)\mathbf{U}^{-1}(t_0)\mathbf{X}(t_0) + \int_{t_0}^{t} \left\{\mathbf{U}(t)\mathbf{U}^{-1}(\tau)\left[\bar{\mathbf{D}}(\tau)\mathbf{U}(\tau) + \bar{\mathbf{E}}(\tau)\,\mathbf{f}(\tau)\right]\right\} \mathrm{d}\tau, \tag{19}$$

in which $\mathbf{U}(t)$ is the fundamental solution matrix and:

$$\dot{\mathbf{U}}(t) = \bar{\mathbf{A}}(t)\mathbf{U}(t), \quad \mathbf{U}(t_0) = \mathbf{I}_{2p}. \tag{20}$$

Also

$$\mathbf{X}(t) = \mathbf{U}(t)\mathbf{X}(t_0). \tag{21}$$

A transfer matrix $\mathbf{\Phi}$ is used to obtain $\mathbf{U}(t)$ such as

$$\mathbf{\Phi}(t,\tau) \approx \mathbf{U}(t)\mathbf{U}^{-1}(\tau). \tag{22}$$

Thus

$$\mathbf{X}(t) = \mathbf{\Phi}(t,\tau)\mathbf{X}(\tau). \tag{23}$$

An approximate solution can be used to obtain $\mathbf{\Phi}$:

$$\mathbf{\Phi}(t_{k+1}, t_k) = \mathrm{e}^{\bar{\mathbf{A}}(t_k)\Delta t_k}, \tag{24}$$

in which $\Delta t_k = t_{k+1} - t_k$ is an assumed time interval. If $\bar{\mathbf{A}}^{-1}(t_k)$ exists, Eq. (18) can easily be solved as follows:

$$\mathbf{X}(t_{k+1}) = \bar{\mathbf{A}}_1(t_k) + \bar{\mathbf{D}}_1(t_k)\mathbf{u}(t_k) + \bar{\mathbf{E}}_1(t_k)\,\mathbf{f}(t_k), \tag{25}$$

where

$$\bar{\mathbf{A}}_1(t_k) \cong \mathrm{e}^{\bar{\mathbf{A}}(t_k)\Delta t_k},$$
$$\bar{\mathbf{D}}_1(t_k) \cong [\bar{\mathbf{A}}_1(t_k) - \mathbf{I}]\bar{\mathbf{A}}^{-1}(t_k)\bar{\mathbf{D}}(t_k),$$
$$\bar{\mathbf{E}}_1(t_k) \cong [\bar{\mathbf{A}}_1(t_k) - \mathbf{I}]\bar{\mathbf{A}}^{-1}(t_k)\bar{\mathbf{E}}(t_k).$$

Solving Eq. (25) with appropriate time step would be more efficient in terms of the computation time needed for a parametric study. Therefore, in this study, the MATLAB program is used to numerically solve Eq. (25).

3. Control algorithm

The applicability of active structural control in reducing the maximum response of a continuous system under the effect of moving mass is investigated using a number of discrete actuators. A linear classical optimal control algorithm with displacement-velocity feedback is used to determine the required control forces. Thus, the following Riccati type matrix equation is considered [16]:

$$\mathbf{P}\,\bar{\mathbf{A}} - \tfrac{1}{2}\mathbf{P}\,\bar{\mathbf{D}}\,\mathbf{R}^{-1}\bar{\mathbf{D}}^{\mathrm{T}}\,\mathbf{P} + \bar{\mathbf{A}}^{\mathrm{T}}\,\mathbf{P} + 2\mathbf{Q} = 0, \tag{26}$$

in which $\mathbf{Q}_{2p\times 2p}$ and $\mathbf{R}_{n\times n}$ are positive semi-definite and positive definite matrices, respectively. The parameters P and p represent the Riccati matrix and the number of controlled modes involved in the control process, respectively, while n is the number of actuators. The resulting control gain matrix becomes:

$$\mathbf{G} = -\tfrac{1}{2}\mathbf{R}^{-1}\,\bar{\mathbf{D}}^{\mathrm{T}}\,\mathbf{P}. \tag{27}$$

Therefore, the control force vector can be shown as

$$\mathbf{u}(t) = \mathbf{G}\mathbf{X}(t). \tag{28}$$

Substituting Eqs. (27) and (28) in Eq. (18) leads to:

$$\dot{\mathbf{X}}(t) = (\bar{\mathbf{A}} + \bar{\mathbf{D}}\,\mathbf{G})\mathbf{X}(t) + \bar{\mathbf{E}}\,\mathbf{f}(t),\ \mathbf{X}(0) = \mathbf{A}_0. \tag{29}$$

Eq. (29) should be solved to determine the controlled response of the system. Also, as Eq. (18) shows, the system matrix $\bar{\mathbf{A}}$ is a function of time, meaning that the control gain matrix components are continuously changing as the moving mass is traveling along the beam. This will change to a constant gain matrix for controlling the free vibration response of the system once the moving mass passes the beam.

4. Numerical examples

In this section, two examples are presented. In the first example, the dynamic response of an uncontrolled simply supported beam under the effect of moving mass and moving load is investigated. Parametric studies are carried out to evaluate the effect of velocity and weight of the moving mass on the beam's response both for the exact and approximate approaches. The importance of the moving mass inertia, as well as the number of vibrational modes needed to accurately determine the beam's response were studied for both approximate and exact formulations. In the second example, the actively controlled response of the same problem has been considered. A parametric study is carried out to investigate the performance of the control system in reducing the dynamic response of the beam under moving mass for different number of vibrational modes and actuators involved in the control process.

4.1. Example 1

A uniform 60 m long, simply supported Euler–Bernoulli beam is considered. The mass per unit length of the beam is 1 kg/m ($m = 1$) and its bending stiffness, EI, is equal to $5.0 \times 10^5\,\mathrm{N\,m^2}$. Any other value can be used for the bending stiffness and the mass per unit length of the beam as long as they lead to the same fundamental period of vibration for the beam. In order to simplify the parametric study, the parameter V' is defined as $V' = 2L/T_p$, in which T_p is the fundamental period of the beam.

First, it is assumed that the beam is under the effect of moving mass and moving load separately considering only the first vibrational mode of the beam. In Fig. 2, the dynamic amplification factor, that is the ratio of beam's absolute maximum dynamic response to its maximum static displacement at the midspan, is plotted versus the velocity of the mass. It should be noted that the maximum dynamic response may happen in the first phase of the motion (while the mass is still on the beam) or in its second phase (the mass has passed through the end point of the beam, and there is a free vibration), depending on the mass velocity. As it is shown, the

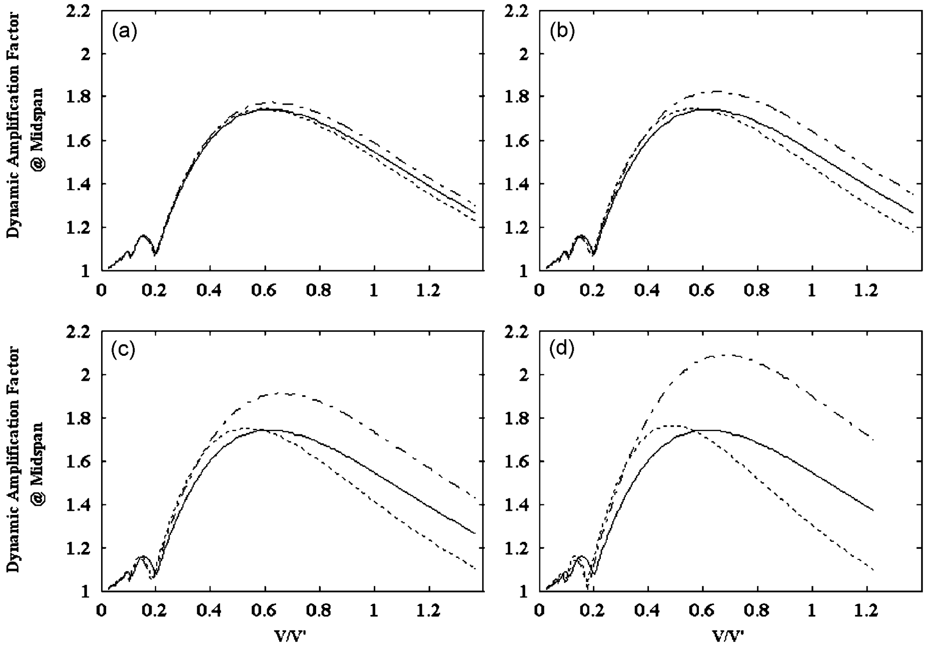

Fig. 2. The effect of inertia due to the changes in velocity and mass: (a) $Mg = 20\,\mathrm{N}$, (b) $Mg = 50\,\mathrm{N}$, (c) $Mg = 100\,\mathrm{N}$, (d) $Mg = 200\,\mathrm{N}$; (—— moving load; ········ moving mass—approximate approach; – · – · – moving mass—exact approach).

inertial effects will be noticeable for the mass velocities larger than $0.2V'$. Also, as Fig. 2 indicates, any increase in the weight of the mass will intensify the inertial effects. Based on this figure, the effect of centripetal and Coriolis accelerations are negligible for the velocities lower than $\approx 0.4V'$ which is named as critical velocity V_{cr}. For velocities larger than that, use of exact formulation will be required.

The proposed critical velocity is fairly large for the common bridge structures due to their larger natural frequencies. It implies that the approximate approach could effectively be used instead of the exact one in most of the practical cases. Moreover, the amount of V_{cr} is slightly decreasing due to increase in mass weight. For example, while the critical velocity is $0.402V'$ for a moving mass equal to $Mg = 20\,\mathrm{N}$, which is about %3 of the beam's whole weight, it decreases to $0.325V'$for a mass equal to $Mg = 200\mathrm{N}$, around %33 of the beam's weight. Also, the variation considered in the weight of the moving mass (up to %33 of the beam's weight) presumably will not violate the basic assumptions made for having a linear behavior in problem formulation. From Eq. (17), one could observe that for a simply supported beam, the mass's velocity should be kept less than $V = V'\sqrt{ml/2M}$ in order to ensure the stability of the dynamic system.

As Fig. 3 shows, increasing the number of considered vibrational modes on the response of the beam mainly improves the accuracy of the approximate approach. In other words, using more number of modes increases the critical velocity, so that the approximate approach can be used instead of the exact one in a larger velocity range of moving mass. Using more than 3 modes does not make any noticeable change in the results for both moving load and moving mass cases.

Equally important, is the effect of change in moving mass weight on the beam's dynamic response for a constant velocity. The results provided in Fig. 4 are for the case with only the first vibrational mode of the system. As Fig. 4 shows, in case of moving load, the maximum response of the beam remains unaffected with respect to any changes in the weight of the moving load. It also shows that for velocities less than critical velocity, the exact and approximate approach lead to very close results that are increasing almost linearly with the weight of moving load.

The effect of including more modes in the formulation for the case $V = 0.5V'$ is shown in Fig. 5. One could see that while the peak response of the beam in the exact approach is slightly reducing, the accuracy of the approximate method in estimating the peak response gets better.

Finally, for a moving mass equal to $Mg = 100\,\mathrm{N}$, Figs. 6 and 7 show the time-history of the midspan deflection of the beam for moving velocities of $V = 0.3V' = 11.1$ and $V = 0.9V' = 33.3\,\mathrm{m/s}$, respectively. As it

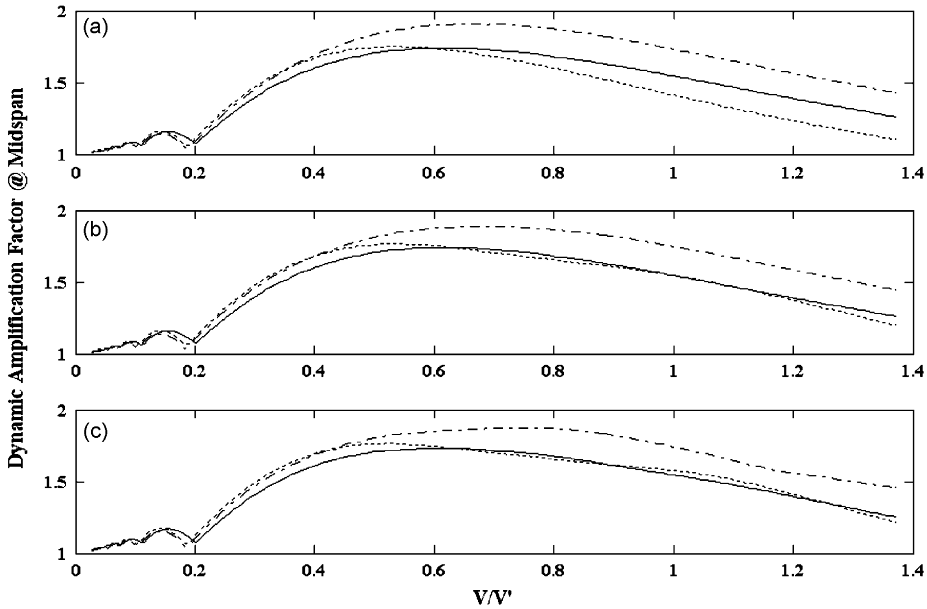

Fig. 3. The effect of considered number of modes for the case $Mg = 100\,\text{N}$: (a) 1 Mode; (b) 2 Modes; (c) 3 Modes. (—— moving load; ········ moving mass—approximate approach; – · – · – moving mass—exact approach).

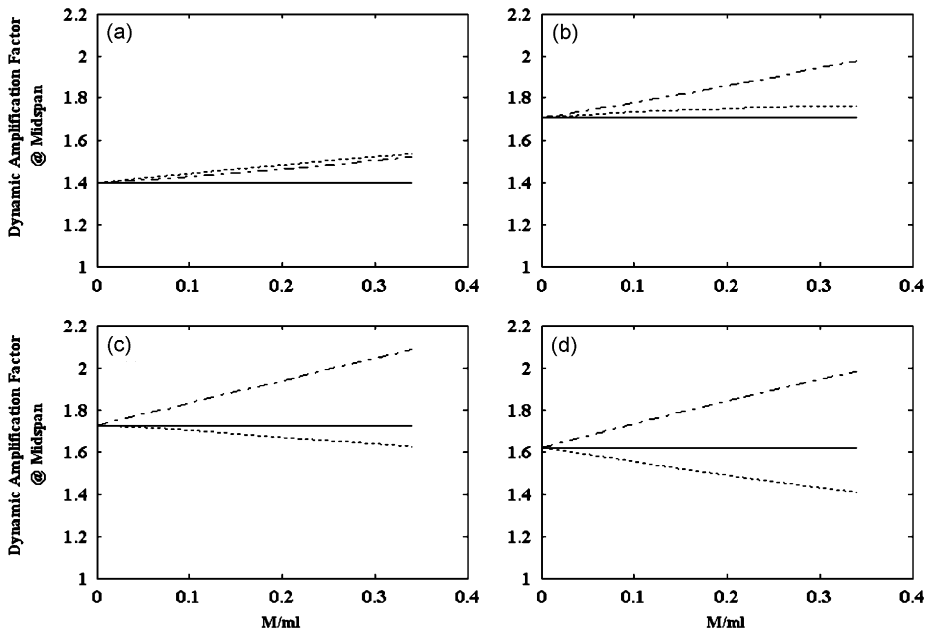

Fig. 4. The effect of change in mass on the response of the beam: (a) $V = 0.3V'$; (b) $V = 0.5V'$; (c) $V = 0.7V'$; (d) $V = 0.9V'$; (—— moving load; ········ moving mass—approximate approach; – · – · – moving mass—exact approach).

was expected, for the mass velocity below the critical one, the response of the beam is almost the same for both approaches and inclusion of higher vibrational modes has no appreciable effect on the results (Fig. 6). The difference between the exact and approximate approaches and the effect of involving higher modes become more important for the moving velocities larger than the critical velocity (Fig. 7).

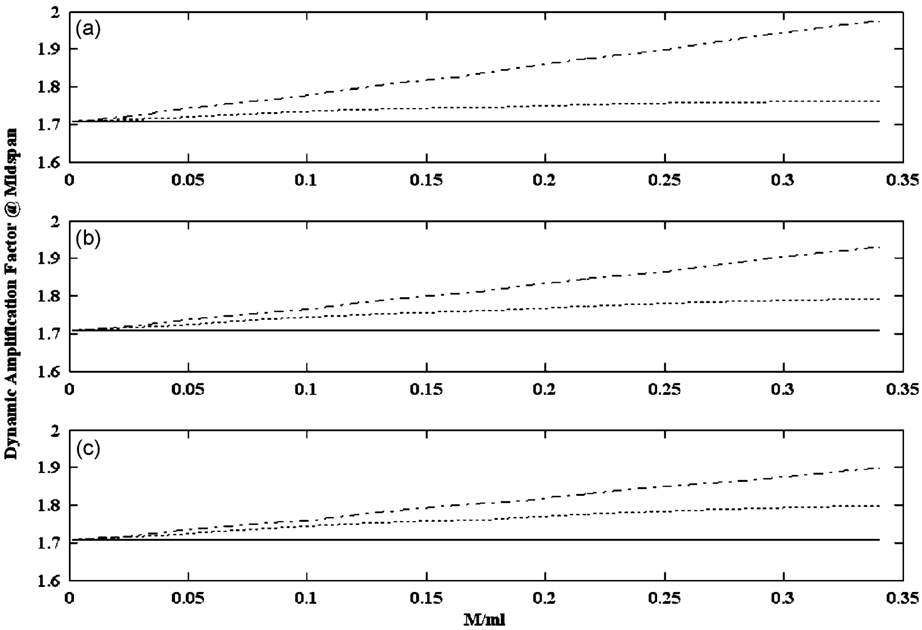

Fig. 5. The contribution of the higher modes to the response of the system for $V = 0.5V'$: (a) 1 Mode; (b) 2 Modes; (c) 3 Modes; (—— moving load; ········ moving mass—approximate approach; – · – · – moving mass—exact approach).

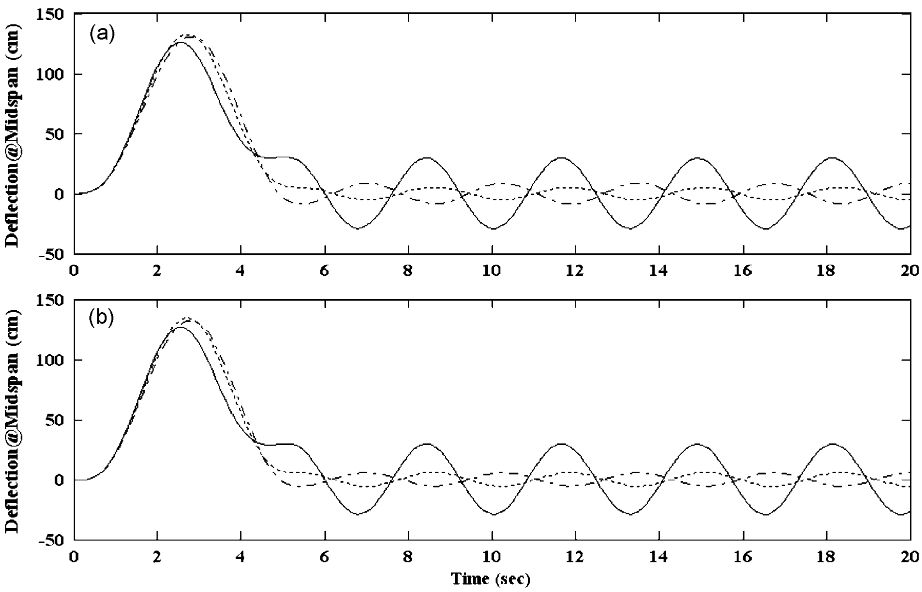

Fig. 6. The time history of the midspan deflection under the effect of moving mass and moving load: (a) 1 Mode; (b) 3 Modes; ($Mg = 100\,\text{N}$, $V = 0.3V' = 11.1\,\text{m/s}$; —— moving load; ········ moving mass—approximate approach; – · – · – moving mass—exact approach).

4.2. Example 2

In this example, the actively controlled response of the previous example for a moving mass velocity equal to $V = 0.3V' = 11.1\,\text{m/s}$ and moving mass weight $Mg = 20\,\text{N}$ is considered. Based on the results of the

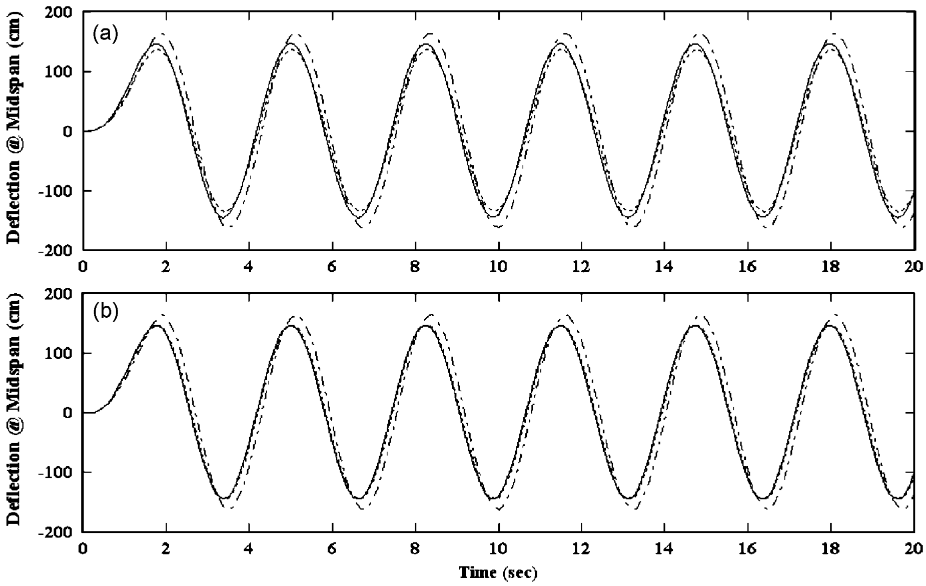

Fig. 7. The time history of midspan deflection under the effect of moving mass and moving load: (a) 1 Mode; (b) 3 Modes; ($Mg = 100\,\mathrm{N}$, $V = 0.9V' = 33.3\,\mathrm{m/s}$; —— moving load; ········ moving mass—approximate approach; – · – · – moving mass—exact approach).

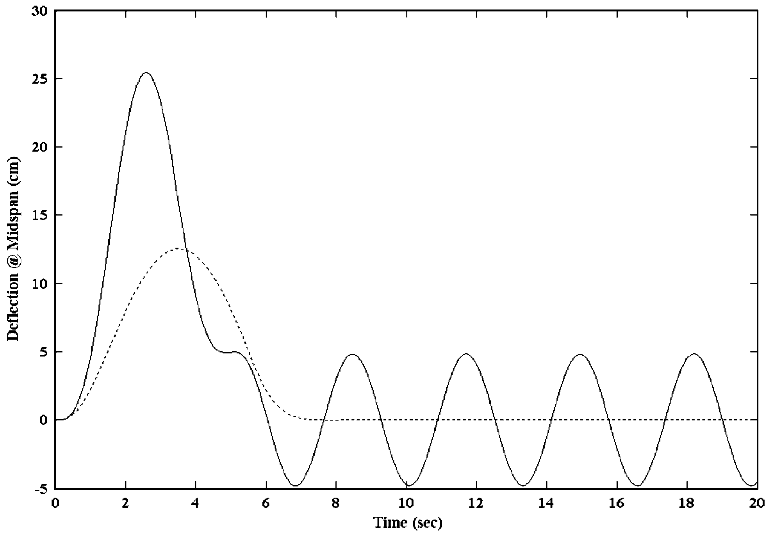

Fig. 8. The time history of the uncontrolled and controlled deflection of the midspan under the moving mass excitation: $Mg = 20\,\mathrm{N}$, $V = 0.3V' = 11.1\,\mathrm{m/s}$; —— uncontrolled; ········ controlled.

previous example, the approximate approach could be effectively used here. Considering one sensor and actuator at midspan and defining the weighting matrices given in Eq. (26) as, $\mathbf{R} = 0.1$ and $\mathbf{Q} = 120\mathbf{I}_{2\times2}$, in which, $\mathbf{I}_{2\times2}$ is an identity matrix, the uncontrolled and controlled response of the beam are shown in Fig. 8. Only the first vibrational mode of the beam is considered here. The control force is adjusted to reduce the maximum deflection of the beam at midspan to half of its uncontrolled response under the effect of moving mass. As it was mentioned before, for the assumed velocity of the moving mass, the maximum response of the beam occurs in the first phase of the motion. The required control force is shown in Fig. 9.

As Fig. 9 indicates, controlling more number of vibrational modes will increase the maximum required control force compared to the first mode control only. One can see the effect of controlling higher modes on the required control forces. When the higher modes are involved, proper values for the weighting matrix $\mathbf{Q}$

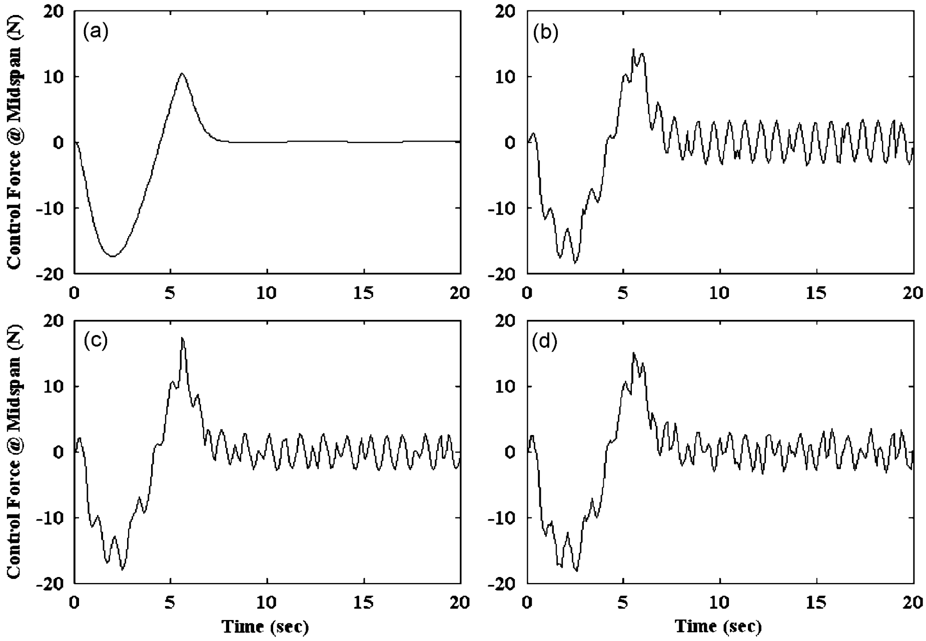

Fig. 9. Required control force at midspan: (a) 1 Mode, $\mathbf{Q} = 120\mathbf{I}_{2\times2}$; (b) 2 Modes, $\mathbf{Q} = 200\mathbf{I}_{4\times4}$; (c) 3 Modes, $\mathbf{Q} = 200\mathbf{I}_{6\times6}$; (d) 5 Modes, $\mathbf{Q} = 240\mathbf{I}_{8\times8}$.

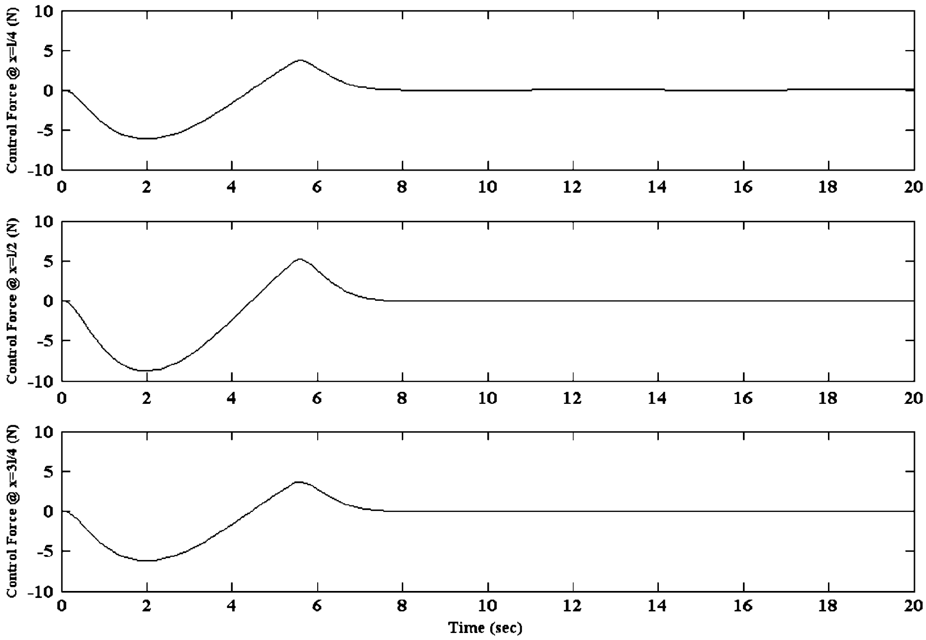

Fig. 10. Required control force for $Mg = 20\,\text{N}$, $V = 0.3V' = 11.1\,\text{m/s}$ at (1/4) spans; 1 Mode.

should be considered in order to achieve the same level of response reduction. No appreciable change occurs in the peak response or necessary control force by including more number of modes in the control process.

In order to study the effect of number of controllers, 3 controllers (actuators) are considered at equal distance along the beam, keeping all the other parameters of the previous part the same. Again, to reduce the

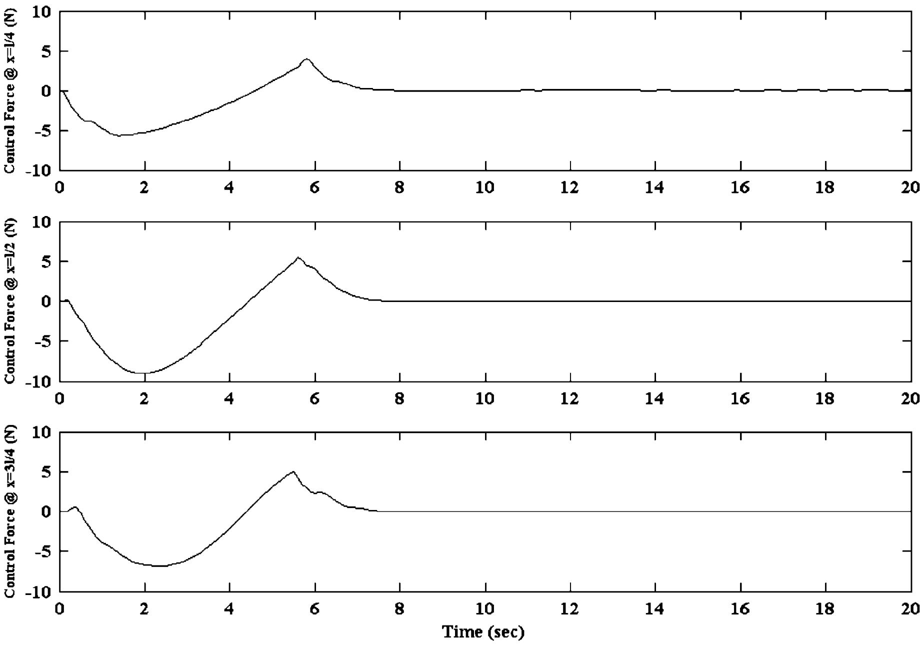

Fig. 11. Required control force for $Mg = 20\,\mathrm{N}$, $V = 0.3V' = 11.1\,\mathrm{m/s}$ at (1/4) spans; 3 Modes.

peak response of the uncontrolled system at midspan in half, the weighting matrices are considered to be $\mathbf{R} = 0.1\mathbf{I}_{n\times n}$ and $\mathbf{Q} = 60\mathbf{I}_{2p\times 2p}$. The required control force for each controller and the effect of controlling more number of modes are shown in Figs. 10 and 11. As expected, the required control forces are reduced by increasing the number of controllers. Moreover, use of multiple actuators enables the control system system to efficiently reduce the contribution of the higher vibrational modes to the response of the system, thus preventing further fluctuation in the applied control force. Also, no significant change obtained in the amount of required control force by including more than 3 modes in the control process.

5. Conclusions

The constitutive equation of an Euler–Bernoulli beam under the excitation of moving mass is considered. The dynamics of the uncontrolled system is governed by a linear, self-adjoint partial differential equation. An approximate formulation to the problem is obtained by limiting the inertial effect of the moving mass merely to the vertical component of acceleration. For a simply supported beam, it is shown that the effect of the centripetal and Coriolis accelerations of the moving mass are negligible for velocities below a critical velocity that is defined in terms of the beam's fundamental period and span. This critical velocity is fairly large compared to those in practical cases, thus the approximate approach can effectively be used instead of exact one for a wide range problems. There is however a slight variation in critical velocity depending on the weight of the moving mass. On the other hand, it is shown that the effect of higher vibrational modes is not negligible for certain velocity ranges. Consideration of the first 3 modes of the system seems to be adequate for getting accurate results especially for high mass velocities. Finally, a linear classical optimal control algorithm with a time varying gain matrix with displacement-velocity feedback is used to control the response of the beam. The efficiency of the control algorithm in suppressing the response of the system under the effect of moving mass with different number of controlled modes and actuators is investigated. Use of multiple actuators enables the control system to efficiently reduce the contribution of the higher vibrational modes to the response of the system, thus preventing further fluctuation in the required control force.

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